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Extremal Problems for the Vector-Valued $\langle L^1/H^1_0, H^\infty \rangle$ Duality

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Let X be a complex Banach space and $L^1(X) := L^1(\mathbb{T}; X)$ the Bochner space on the circle $\mathbb{T}$. The X-valued Hardy space $\mathbb{H}^1_0(X) := \{f \in L^1(X) : \hat{f}(n) = 0 \ \forall n \leqslant 0\}$ is proximinal in $L^1(X)$ if H has ARNP and is contractively complemented in X''. It is semi-Chebyshev if X is strictly convex. With $H^\infty(X')$ the dual space of $L^1(X)/\mathbb{H}^1_0(X)$, extremal kernels and functions for this duality are studied. Proximinality fails for $X := L^1/H^1_0$; this is equivalent to the assertion that for $\Lambda := \mathbb{N} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{N}$, $L^1_\Lambda(\mathbb{T}^2)$ is not proximinal in $L^1(\mathbb{T}^2)$. A class of subsets $\Lambda \subset \mathbb{Z}^2$ is described for which this non-proximinality holds. © 1996 Academic Press, Inc.

1. Introduction and Preliminaries

A classical theorem first proved in 1941 by Doob [Do, Theorem 3] and reproved by many authors (Khavinson, see [Kh, 9.]; Rogosinski and Shapiro [RS]; Pták, see [Khe]) states that $H^1_0 := \{f \in L^1(\mathbb{T}) : \hat{f}(n) = 0 \ \forall n \leqslant 0\}$ is a Chebyshev subspace of $L^1 = L^1(\mathbb{T})$, the Lebesgue space of the circle group $\mathbb{T}$. (A subset A of a metric space M is called semi-Chebyshev resp. proximinal if for every $x \in M$ at most resp. at least one best approximation in A exists, and Chebyshev if proximinal and semi-Chebyshev, see [Si2, Definitions 2.1, 3.1].) In other words, every coset in L^1/H^1_0 contains exactly one representative of the least possible (= coset) norm. Taking into account the duality $(L^1/H^1_0)' = H^\infty := \{h \in L^\infty(\mathbb{T}) : \hat{h}(n) = 0 \ \forall n < 0\}$, this is the solution of the following "extremal problem": given a "kernel" $f \in L^1$, consider the functional $h \mapsto \int_\mathbb{T} fh \, d\lambda$ on H^∞ (λ = Haar measure) and find uniquely an "equivalent" kernel $f_0 \in L^1$ (i.e. giving the same functional, $\Leftrightarrow f_0 - f \in H^1_0$) with $\|f_0\|_1$ realizing the functional (= coset) norm. Such an f_0 is called an extremal kernel. A "dual extremal function" is a function $h \in H^\infty$ with $\|h\|_\infty \leqslant 1$ and $\int fh \, d\lambda$ realizing the functional norm; it also exists uniquely. This theory is presented in detail in the books of Duren [Du, Chapter 8] and Garnett [Ga, IV], and the first aim of this article,

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mainly part of the author's habilation thesis [H3], is to give a vector-valued generalization (§2). In §3 I discuss an example leading to a proximinality problem in two-variable (scalar-valued) Fourier analysis which might be of independent interest. A short summary of results is postponed to the end of this section. For more details on the preliminaries the reader is referred to [H3].

1.1. *Spaces* $L^1(X)$ *and* $M(X)$. Let X be a complex Banach space (dual X', unit ball B_X), then $L^1(X) = L^1(\lambda; X)$ denotes the usual Lebesgue-Bochner space [DU, II] and $M(X) = M(\Sigma; X)$ the space of (σ-additive) X-valued measures of bounded variation defined on the Borel σ-algebra Σ of $\mathbb{T}$. Under the variation norm on $M(X)$, $L^1(X) \subset M(X)$ isometrically via $f \mapsto f \cdot \lambda$, and Singer's theorem [Si 1, pp. 398ff.] states that $(f, m) \mapsto \int \langle f, dm \rangle$ is a dual pairing on $C(\mathbb{T}; X) \times M(\Sigma; X')$ rendering $M(\Sigma; X')$ the dual space of $(C(\mathbb{T}; X), \|\cdot\|_\infty)$.

1.2. $L^\infty(X', X)$, *the Dual of* $L^1(X)$. A function $f\colon \mathbb{T} \to X'$ is called weak*-λ-measurable, if $\forall x \in X$ the function $\langle x, f \rangle\colon \mathbb{T} \to \mathbb{C}$ is λ-measurable. For such a function, there exists $|f| := \sup_{x \in B_X} |\langle x, f \rangle|$, the supremum being taken in the order-complete vector lattice $L^0(\lambda; \mathbb{R})$ of λ-measurable functions modulo λ-null functions [KA, p. 42f.]. Note that $|f|(t) \leqslant \|f(t)\|_{X'}$ a.e. and the inequality may be strict. However, if X is separable, *or* if f is strongly measurable, then $|f|$ equals $\|f(\cdot)\|$ a.e. (see [H3, 1.3]).

Define $\mathscr{L}^\infty(\lambda; X', X) := \{f\colon \mathbb{T} \to X' \text{ weak* measurable: } |f| \in L^\infty(\lambda)\}$ equipped with the seminorm $\|f\|_\infty := \|\, |f| \,\|_\infty$, and finally $L^\infty(X', X) = L^\infty(\lambda; X', X) := \mathscr{L}^\infty(\lambda; X', X) / \|\cdot\|_\infty^{-1}(0)$.

For $f \in L^1(X)$, $g \in L^\infty(X', X)$, the function $\langle f(\cdot), g(\cdot) \rangle =: \langle f, g \rangle$ is a well-defined (!) member of L^1 and $|\langle f, g \rangle| \leqslant |f| |g|$ a.e. [H1, (0.5) 5^0]. Under the pairing $(f, g) \mapsto \int \langle f, g \rangle \, d\lambda$ on $L^1(X) \times L^\infty(X', X)$, the space $L^\infty(X', X)$ becomes the dual of $L^1(X)$ (Bukhvalov [B1, Theorem 0.1], Ionescu-Tulcea [IT, VII.4 Theorem 7, Corollary], Schwartz [Sc, Corollaire (2.3)], see also [DS, VI.8.7]).

1.3. *Hardy Spaces*, *ARNP*. As a general notation, if $E(\cdot)$ is any of the spaces $L^1(X)$, $L^\infty(X', X)$, or $M(X)$ and if $\Lambda \subset \mathbb{Z}$, then $E_\Lambda(\cdot)$ is the subspace of members of $E(\cdot)$ whose Fourier coefficients vanish off Λ. (If $E(\cdot) = L^\infty(X', X)$, the integral defining the coefficients is the Gel'fand or weak* integral [DU, p. 53].) The Hardy spaces in this article are $\mathbb{H}_0^1(X) := L^1_{\mathbb{N}}(X)$ resp. $H^\infty(X') := L^\infty_{\mathbb{N}_0}(X', X)$. The Banach space X has the analytic Radon-Nikodým property ARNP introduced by Bukhvalov and Danilevich [B1], [BD], [H1] iff $\mathbb{H}_0^1(X) = M_{\mathbb{N}}(X)$ (see [H1, (2.10)]). The basic examples of ARNP spaces are RNP spaces (because $m \in M_{\mathbb{N}}(X) \Rightarrow m \ll \lambda$ by the F. and M. Riesz theorem [Du, Theorem 3.8]), weakly sequentially complete

Banach lattices [BD, Theorem 3], [H 1, (3.5)], and preduals of von Neumann algebras [HP, 2.3].

1.4. *$H^\infty(X')$ Is the Dual of $L^1(X)/\mathbb{H}_0^1(X)$.* Modulo general Banach space theory [Du, Theorem 7.2] this amounts to saying that $H^\infty(X') \subset L^\infty(X', X) = L^1(X)'$ is the annihilator of $\mathbb{H}_0^1(X) \subset L^1(X)$. Clearly the annihilator is contained in $H^\infty(X)'$. Conversely, let $f \in \mathbb{H}_0^1(X)$, $g \in H^\infty(X')$ be given.

CLAIM. *$\langle f, g \rangle \in H_0^1$ (in particular $\int \langle f, g \rangle \, d\lambda = 0$).*

Proof. Let $P_r(t) := \sum_{n \in \mathbb{Z}} r^{|n|} e^{int}$ be the Poisson kernel, then as in the scalar case, $P_r * f \to f$ $(r \to 1)$ in $L^1(X)$ [B1, Theorem 2.1], [H1, Satz (1.11)], f being strongly measurable. This implies $\forall n \in \mathbb{Z}$:

$$\langle P_r * f, g \rangle^\wedge (n) = \int \langle P_r * f(t), e^{-int} g(t) \rangle \, \lambda(dt) \to \langle f, g \rangle^\wedge (n) \ (r \to 1).$$

By [H2, 4.2] the integral equals $\sum_{k=1}^\infty r^k \langle \hat{f}(k), \hat{g}(n-k) \rangle = 0$ if $n \leqslant 0$. ∎

1.5. *Summary.* $\mathbb{H}_0^1(X)$ lies proximinal in $L^1(X)$ if X has ARNP and is norm-1 complemented in the bidual X'' (2.1). The usual characterization of extremal kernels and functions is given (2.3). The former is unique if X is strictly convex, the latter if X is smooth and an extremal kernel exists (2.5). In §3 it is shown that for $X = L^1/H_0^1$ (which fails ARNP) $\mathbb{H}_0^1(X)$ is not proximinal in $L^1(X)$. This turns out to be equivalent to the assertion that for $\Lambda := \mathbb{N} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{N} \subset \mathbb{Z}^2$ the space $L_\Lambda^1(\mathbb{T}^2)$ is not proximinal in $L^1(\mathbb{T}^2)$. The proof of this assertion (in fact, of a general criterion 3.3) consists of a reduction to the fact stated by Kahane that if $\Gamma \subset \mathbb{Z}$ with $1 < \# \mathbb{Z} \setminus \Gamma < \infty$ then $L_\Gamma^1(\mathbb{T})$ is not proximinal in $L^1(\mathbb{T})$. Since these seem to be the only known examples of non-proximinal translation-invariant subspaces of $L^1(\mathbb{T})$ (see [Ka]), this criterion might be interesting in itself.

2. VECTOR-VALUED THEORY

The theory developped in this section has useful applications in the study of weak compactness in $L^1(X)/\mathbb{H}_0^1(X)$ [H3, 3.6].

2.1. THEOREM. *If X has ARNP and is complemented in X'' by a contractive projection then $\mathbb{H}_0^1(X)$ lies proximinal in $L^1(X)$.*

The hypotheses are satisfied e.g. if X is a separable (or RNP) dual space, or a weakly sequentially complete Banach lattice, or a predual of a von Neumann algebra (1.3), [LT, 1.c.4], [T, III.2.14].

Proof. $M_{\mathbb{N}}(X'')$ is obviously closed in $M(X'')$ for the weak* topology $\sigma(M(X''), C(X'))$ (1.1), hence proximinal [Si2, Theorem 2.9]. Considering (see 1.1) $L^1(X)$ as a subspace of $M(X) \subset M(X'')$, for a fixed $f \in L^1(X)$ there exists $m_0'' \in M_{\mathbb{N}}(X'')$ with $\|f - m_0''\| \leqslant \|f - m''\|$, all $m'' \in M_{\mathbb{N}}(X'')$.

Let $P\colon X'' \to X$ be a projection of norm 1, as assumed, and $m_0 := P \circ m_0'' \in M(X)$; clearly $m_0 \in M_{\mathbb{N}}(X)$. By the ARNP hypothesis, $m_0 = h_0 \cdot \lambda$ for some $h_0 \in \mathbb{H}_0^1(X)$. Then for $h \in \mathbb{H}_0^1(X) \subset M_{\mathbb{N}}(X'')$, $\|f - h\|_1 \geqslant \|f - m_0''\| \geqslant \|P \circ (f - m_0'')\| = \|f - h_0\|_1$. ∎

This proof is certainly the simplest (the slightly different approach of [Kh, 9.], [Du, p. 130 f.] would also work).

Extremal Kernels and Functions

Consider the dual pairing $\langle L^1(X)/\mathbb{H}_0^1(X), H^\infty(X')\rangle$ which is of the form $\langle Z, Z'\rangle$ (1.4) and thus mutually norming. Fix $f \in L^1(X) \backslash \mathbb{H}_0^1(X)$; the coset $f + \mathbb{H}_0^1(X)$ of f in the quotient space will be denoted by $[f]$ in the sequel; $[f] \neq 0$. Following Rogosinski, Shapiro [RS] and Duren [Du, Chapter 8],

2.2. Definition. (1) $f_0 \in L^1(X)$ is called an *extremal kernel* for $[f]$ if $f_0 \in [f]$ and $\|f_0\|_1 = \|[f]\|$;

(2) $h_0 \in H^\infty(X')$ is called a (*dual*) *extremal function* for $[f]$ if $\int \langle f, h_0\rangle \, d\lambda = \|[f]\|$ and $\|h_0\|_\infty \leqslant 1$.

Thus, f_0 should be an element of smallest norm in $[f]$ whereas h_0 is required to be a support functional of $[f]$.

Under the hypotheses of 2.1, f_0 exists for every $[f]$. I will prove in §3 that without ARNP f_0 need not exist. On the other hand, h_0 of course always exists by Hahn-Banach.

A characterization of the following type looks familiar in the theory of extremal problems.

2.3. Proposition. *Let $f_0 \in [f]$ and $h_0 \in B_{H^\infty(X')}$. Then f_0 is an extremal kernel and h_0 a dual extremal function for $[f]$ if and only if $\langle f_0(t), h_0(t)\rangle = \|f_0(t)\|$* a.e. *In this case, $|h_0|(t) = 1$ a.e. where $f_0(t) \neq 0$.*

Proof. "only if" By hypothesis,

$$\int |f_0| \, d\lambda = \|f_0\|_1 = \|[f]\| = \int \langle f_0, h_0\rangle \, d\lambda$$

and $\|h_0\|_\infty \leqslant 1$; by 1.2, $|\langle f_0, h_0\rangle| \leqslant |f_0| |h_0| \leqslant |f_0|$ a.e., thus $\langle f_0, h_0\rangle = |f_0|$ a.e.

"if"

$$\|f_0\|_1 = \int |f_0| \, d\lambda = \int \langle f_0, h_0 \rangle \, d\lambda$$

$$= \int \langle f, h_0 \rangle \, d\lambda \leqslant \|[f]\| \, \|h_0\|_\infty \leqslant \|[f]\| \leqslant \|f_0\|_1.$$

Last assertion: By 1.2 again, $\|f_0(t)\| = \langle f_0(t), h_0(t) \rangle \leqslant \|f_0(t)\| \, |h_0|(t) \leqslant \|f_0(t)\|$ a.e. and the last statement follows. ▮

2.4. COROLLARY. *If X has ARNP and is contractively complemented in X'' then the set $\{h \in H^\infty(X') \colon |h|(t) = \|h\|_\infty$ on a set of positive measure$\}$ is norm dense in $H^\infty(X')$.*

Proof. Let $h \in H^\infty(X') = (L^1(X)/\mathbb{H}_0^1(X))'$ attain its norm $\|h\|_\infty$ as a functional on $L^1(X)/\mathbb{H}_0^1(X)$ in some $[f] \in L^1(X)/\mathbb{H}_0^1(X)$, $\|[f]\| = 1$. I claim that $|h|(t) = \|h\|_\infty$ on a set of positive measure; the assertion then follows from the Bishop–Phelps theorem [Di, p. 3]. For the claim, I can assume w.l.o.g. $\|h\|_\infty = 1$. But then h is an extremal function for $[f]$. Choose an extremal kernel f_0 for $[f]$; this is possible by theorem 2.1. Now by 2.3, $|h|(t) = 1 = \|h\|_\infty$ a.e. on the set of positive measure $\{f_0 \neq 0\}$. ▮

It can be proved that the conclusion of this corollary (due to Fisher [F, p. 482] in the scalar case) holds also under the (incomparable) assumption that X' has ARNP [H3, Corollary 2.12].

2.5. THEOREM (Uniqueness). (1) *Let $f_1, f_2 \in L^1(X)$ be extremal kernels for $[f]$. Then*

(a) $\|f_1(t)\| = \|f_2(t)\|$ a.e.

(b) *If X is strictly convex then even $f_1 = f_2$ in $L^1(X)$.*

(2) *Let $h_1, h_2 \in H^\infty(X')$ be extremal functions for $[f]$. Suppose that there exists (at least) one extremal kernel f_0 for $[f]$. Then*

(a) $|h_1|(t) = |h_2|(t) = 1$ *a.e. where* $f_0(t) \neq 0$.

(b) *If X is smooth then even $h_1 = h_2$ in $H^\infty(X')$. (A Banach space is called "smooth" if every point $\neq 0$ has a unique support functional.)*

Proof. (1.a) Choose an extremal function $h_0 \in B_{H^\infty(X')}$ for $[f]$. By the proposition, $\langle f_1(t) - f_2(t), h_0(t) \rangle = \|f_1(t)\| - \|f_2(t)\| \in \mathbb{R}$ a.e. Since the lefthand side is a member of H_0^1 (1.4) is must be 0 a.e.

(b) It remains to prove $f_1(t) = f_2(t)$ a.e. where $f_1(t) \neq 0 \neq f_2(t)$. For those t, it follows from $\langle f_1(t), h_0(t) \rangle = \|f_1(t)\| = \|f_2(t)\| = \langle f_2(t), h_0(t) \rangle$ a.e. that $f_1(t) = f_2(t)$ a.e., by strict convexity of X.

(2.a) See Proposition 2.3.

(b) Again by Proposition 2.3, $\langle f_0(t), h_1(t)\rangle = \|f_0(t)\| = \langle f_0(t), h_2(t)\rangle$ a.e. Since $f_0 \neq 0$ and X is smooth, this implies $h_1(t) = h_2(t)$ on a set of positive measure (for fixed representatives of $h_1, h_2 \in H^\infty(X') \subset L^\infty(X', X)$). Therefore, if $x \in X$ is also fixed, $\langle x, h_1(t)\rangle = \langle x, h_2(t)\rangle$ a.e., by the identity theorem for H^∞ [Du, Theorem 2.2]. This is the assertion. ∎

Remarks. (i) Trivial (two-dimensional) examples show that neither extremal kernel nor function need be unique in general.

(ii) Part (1.b) says in other words that $\mathbb{H}^1_0(X)$ is a semi-Chebyshev subspace of $L^1(X)$ if X is strictly convex.

2.6. COROLLARY. *If X has ARNP and is complemented in X'' by a contractive projection then $L^1(X)/\mathbb{H}^1_0(X)$ is smooth if (and only if) X is smooth.*

Proof. Combine 2.5 (2.b) with 2.1. (The "only if" assertion is trivial since X can be identified with a subspace of $L^1(X)/\mathbb{H}^1_0(X)$.) ∎

3. EXAMPLE AND A TWO-VARIABLE RESULT

Without the ARNP assumption on X, $\mathbb{H}^1_0(X)$ need not be proximinal in $L^1(X)$. Although this is not particularly surprising, I am working out an example of this phenomenon, because I found the blend of harmonic analysis and approximation theory needed to establish it rather appealing. Note that L^1/H^1_0 fails ARNP [BD, Proposition 4.3], [H1, 3.3] and is contractively complemented in the bidual [A, Theorem 2], [Go1, p. 229 f].

3.1. EXAMPLE. For $X := L^1/H^1_0$, $\mathbb{H}^1_0(X)$ is *not* proximinal in $L^1(X)$.

This assertion is established in several steps. First, it is reduced to a scalar problem in two variables, then further reduced to a minimal extrapolation problem in one variable the answer to which is known. To begin with, I have to consider several natural identifications in which $\hat{\otimes}$ denotes the projective tensor product (see [DU, VIII.1.10], [Kö, §41.5(8)], [DS, III.11.16, 17] for justification): For $X := L^1/H^1_0$, $L^1(X) = L^1 \hat{\otimes} L^1/H^1_0 = (L^1 \hat{\otimes} L^1)/(\overline{L^1 \otimes H^1_0}) = L^1(\mathbb{T}^2)/L^1_{\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)$. Let $f \in L^1(\mathbb{T}^2)$ and $F \in L^1(X)$ the element corresponding to $f + L^1_{\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)$ under this chain of identifications. Then for $m \in \mathbb{Z}$, $X = L^1/H^1_0 \ni \hat{F}(m)$ $= \int_0^{2\pi} f(s, \cdot)\, e^{-ims}\, ds/2\pi + H^1_0$, so that $\hat{F}(m) = 0 \in L^1/H^1_0 \Leftrightarrow \forall n \leqslant 0$: $\int_0^{2\pi}\int_0^{2\pi} f(s,t)\, e^{-ims} e^{-int} (ds/2\pi)(dt/2\pi) = 0$. Altogether, $F \in \mathbb{H}^1_0(X) \Leftrightarrow \hat{F}(m) = 0$ $\forall m \leqslant 0 \Leftrightarrow \forall m \leqslant 0\ \forall n \leqslant 0 : \hat{f}(m,n) = 0 \Leftrightarrow f \in L^1_{\mathbb{N}\times\mathbb{Z}\cup\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)$. In other words, under the identification $L^1(X) = L^1(\mathbb{T}^2)/L^1_{\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)$ above, the subspace $\mathbb{H}^1_0(X)$ identifies with $L^1_{\mathbb{N}\times\mathbb{Z}\cup\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)/L^1_{\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)$. Now I use a result of

Cheney and Wulbert [Si2, Theorem 2.20], saying that if $U \subset V \subset W$ is a chain of inclusions of Banach spaces, where U is proximinal in W, then V is proximinal in W if (and only if) V/U is proximinal in W/U. Of course, I want to apply this lemma to the chain of inclusions $L^1_{\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2) \subset L^1_{\mathbb{N}\times\mathbb{Z}\cup\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2) \subset L^1(\mathbb{T}^2)$. This can be done because of the following proposition.

3.2. Proposition. *$L^1_{\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)$ is proximinal in $L^1(\mathbb{T}^2)$.*

Proof. One of the applications of the Bukhvalov–Lozanovskii theorem says that a subspace $Z \subset L^1(\mu)$ is proximinal if B_Z is closed for the topology of convergence in measure μ (a finite measure) [BL, Theorem 1.6′], [B2, Theorem 1.7], [KA, X.5 Theorem 5]; such a Z is called "nicely placed" in $L^1(\mu)$ by Godefroy [Go1], [Go2] who observed that H^1_0 is nicely placed in $L^1(\mathbb{T})$ [Go1, p. 230]. Therefore, it suffices to establish that $L^1_{\mathbb{Z}\times\mathbb{N}}(\mathbb{T}^2)$ is nicely placed in $L^1(\mathbb{T}^2)$. This is a special case of [Go2, Theorem 2.7], or of the observation stated in [HWW, p. 200]: if Z is nicely placed in $L^1(\mathbb{T})$ then $L^1(\mathbb{T}; Z)$ is nicely placed in $L^1(\mathbb{T}^2)$. (A third proof is given in [H3, p. 46]). ∎

By now it is established that the assertion of example 3.1 is equivalent to

3.3. Example. $L^1_{\mathbb{Z}\times\mathbb{N}\cup\mathbb{N}\times\mathbb{Z}}(\mathbb{T}^2)$ is *not* proximinal in $L^1(\mathbb{T}^2)$.

This is obviously a special case of the following criterion. An affine straight line in $\mathbb{Z}^2$ is a set of the form $\Delta = \mathbb{Z}(m_1, n_1) + (m_0, n_0)$ for *relatively prime* $m_1, n_1 \in \mathbb{Z}$ and arbitrary $m_0, n_0 \in \mathbb{Z}$; with $a = -n_1$, $b = m_1$, $c = am + bn_0$, Δ has also the description $\Delta = \{(m, n) \in \mathbb{Z}^2 : am + bn = c\}$.

Theorem. *Let $\Lambda \subset \mathbb{Z}^2$, and suppose there exists an affine straight line $\Delta \subset \mathbb{Z}^2$ which intersects $\complement\Lambda$ in finitely many but at least 2 points. Then $L^1_\Lambda(\mathbb{T}^2)$ is not proximinal in $L^1(\mathbb{T}^2)$.*

Proof. I reduce the question to a one-variable problem by showing that if $L^1_\Lambda(\mathbb{T}^2)$ were proximinal in $L^1(\mathbb{T}^2)$ then $L^1_\Gamma(\mathbb{T})$ would have to be proximinal in $L^1(\mathbb{T})$ for $\Gamma = \mathbb{Z} \setminus \Gamma'$, $2 \leqslant \#\Gamma' < \infty$, which is known to be false [Ka, p. 303, 3.]. The idea of this reduction is easy: embed $L^1(\mathbb{T})$ into $L^1(\mathbb{T}^2)$ "along Δ". That is, with the data describing Δ as above, define $J: L^1(\mathbb{T}) \to L^1(\mathbb{T}^2)$, $Jf(s, t) := e^{i(m_0 s + n_0 t)} f(m_1 s + n_1 t)$. J is a (well-defined!) isometric embedding onto $L^1_\Delta(\mathbb{T}^2)$, and for $(m, n) = k(m_1, n_1) + (m_0, n_0) \in \Delta$ ($k \in \mathbb{Z}$) the relation $\widehat{Jf}(m, n) = \hat{f}(k)$ holds ($f \in L^1(\mathbb{T})$). Now let $\Gamma := \{k \in \mathbb{Z} : k(m_1, n_1) + (m_0, n_0) \in \Lambda\} = \mathbb{Z} \setminus \Gamma'$ with $2 \leqslant \#\Gamma' < \infty$ by hypothesis. Then J maps $L^1_\Gamma(\mathbb{T})$ onto $L^1_{\Delta\cap\Lambda}(\mathbb{T}^2) = L^1_\Delta(\mathbb{T}^2) \cap L^1_\Lambda(\mathbb{T}^2)$. The situation is as follows:

$$\begin{array}{ccc} L^1_\Lambda(\mathbb{T}^2) & \subset & L^1(\mathbb{T}^2) \\ \cup & & \cup \\ L^1_{\Delta\cap\Lambda}(\mathbb{T}^2) & \subset & L^1_\Delta(\mathbb{T}^2) \\ \wr\big\uparrow J & & \wr\big\uparrow J \\ L^1_\Gamma(\mathbb{T}) & \subset & L^1(\mathbb{T}) \end{array}$$

3.4. LEMMA. *Consider the following diagram of Bananch spaces and inclusions*:

$$\begin{array}{ccc} Y_2 & \subset & X_2 \\ \cup & & \cup \\ Y_2\cap X_1 =: Y_1 & \subset & X_1 \end{array}$$

Suppose there exists a contractive projection $X_2 \twoheadrightarrow X_1$ *leaving* Y_2 *invariant. Then* Y_2 *proximinal in* X_2 *implies* Y_1 *proximinal in* X_1.

Proof. Trivial. ∎

The proof of Theorem 3.3 is finished by establishing

3.5. LEMMA. *There is a contractive projection* $L^1(\mathbb{T}^2) \twoheadrightarrow L^1_\Delta(\mathbb{T}^2)$ *leaving* $L^1_\Lambda(\mathbb{T}^2)$ *invariant.*

Proof. Define $\mu \in M(\mathbb{T}^2) = C(\mathbb{T}^2)'$, $\mu(f) := \int_0^{2\pi} e^{ic\vartheta} f(a\vartheta, b\vartheta)\, d\vartheta/2\pi$, then $\|\mu\| \leqslant 1$ and $\hat{\mu} = 1_\Delta$. The sought projection is $f \mapsto \mu * f$. ∎

3.6. *Note.* It has been said that if $\Gamma = \mathbb{Z}\backslash\Gamma'$, $2 \leqslant \#\Gamma' < \infty$, then it is known [Ka, p. 303, 3.] that $L^1_\Gamma(\mathbb{T})$ is not proximinal in $L^1(\mathbb{T})$. In other words (see [loc.cit.]), the following discrete version of Beurling's "minimal extrapolation problem" does not always have a solution: Given a function φ on Γ' of the form $\varphi = \hat{f} \mid \Gamma'$ for some $f \in L^1(\mathbb{T})$, find a $g \in L^1(\mathbb{T})$ of smallest norm with $\hat{g} \mid \Gamma' = \varphi$. Kahane [loc.cit.] claims that $\varphi := 1$ has no such "minimal extrapolation" g. To see this, by an argument involving an approximate identity for $L^1(\mathbb{T})$ (e.g., $(P_r)_{r<1}$, see [Sh, 7.3.6]) it is proved first that $\|g\|_1 = 1$, then assuming w.l.o.g. that $0 \in \Gamma'$, it follows that $g \geqslant 0$ whence $\hat{g}$ is positive definite. However, by [R, 1.4.1(4^0)] any positive definite function Ψ on $\mathbb{Z}$ with $\Psi(0) = \Psi(k) = 1$ for some $k \neq 0$ must be k-periodic. Applied to $\hat{g}$, this contradicts the Riemann-Lebesgue lemma.

Note added in proof. Q. Xu observed that Proposition 3.2 follows also from Theorem 2.1, taking $X := L^1(\mathbb{T})$ there.

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